Chapter 2

Fuzzy Logic

The chapter gives first a short description of classical and many-valued logics. Classical (two-valued) logic deals with propositions that are either true or false. In many-valued logic, a generalization of the classical logic, the propositions have more than two truth values. Fuzzy logic is an extension of the many-valued logic in the sense of incorporating fuzzy sets and fuzzy relations as tools into the system of many-valued logic. Fuzzy logic provides a methodology for dealing with linguistic variables and describing modifiers like very, fairly, not, etc. Fuzzy logic facilitates common sense reasoning with imprecise and vague propositions dealing with natural language and serves as a basis for decision analysis and control actions.

2.1 Basic Concepts of Classical Logic

Here, some basic concepts of the classical[1] (mathematical) or two-valued logic are briefly reviewed.

Propositions

A *proposition*, also called *statement*, is a declarative sentence that is logically either *true* (T) denoted by 1 or *false* (F) denoted by 0. The set $T_2 = \{0, 1\}$ is called *truth value set* for the proposition. In other words a proposition may be considered as a quantity which can assume one of two values: *truth* or *falsity*.

Example 2.1

Consider the sentences:

(a) The stock market is independent of inflation rates (false proposition);

(b) Money supply is an economic indicator (true proposition);

(c) The price of a product is x dollars where $x > 100$ (contains a variable; neither true nor false, it is not a proposition);

(d) Is the stock market going up? (it is not a proposition).

□

We use letters, $p, q, r, \ldots$, to represent propositions.

The propositions (a) and (b) in Example 2.1 are *simple.*

Compound propositions consist of two or more simple propositions joined by one or more *logical connectives.*

Consider the propositions p and q whose truth values belong to the truth value set $\{0, 1\}$. The meaning of the logical connectives is given by definitions and expressed by equations in which p and q stand for the truth values of the propositions p and q.[2]

Negation

Negation or *denial* of p, denoted $\overline{p}$ (read *not* p) is true when p is false and vice versa, hence

$$\overline{p} = 1 - p. \tag{2.1}$$

Conjunction

Conjunction of p and q, denoted $p \wedge q$ (read p *and* q) is true when p and q are both true (*and* is the common and in English);

$$p \wedge q = \min(p, q). \tag{2.2}$$

Disjunction

Disjunction of p and q, denoted $p \vee q$ (read p *or* q) is true when p or q is true or both p and q are true;

$$p \vee q = \max(p, q). \tag{2.3}$$

Implication (*Conditional proposition*)

The proposition p *implies* q, denoted $p \to q$ (also read *if* p *then* q) is true except when p is true and q is false; p and q are called *premise* (*antecedent*) and *conclusion* (*consequent*) , correspondingly;

$$p \to q = \min(1, 1 + q - p). \tag{2.4}$$

It should be emphasized that the truth or falsity of a compound proposition (formulas (2.1)–(2.4)) is determined only by the truth values of its simpler propositions p and q.

Truth tables

A very useful device to deal with the truth values of compound propositions is the truth table.[3]

The truth values of the operations (2.1)–(2.4) under all possible truth value for p and q are presented in Table 2.1 (1 stands for truth(T) and 0 for false(F)). The right hand sides of (2.1)–(2.4) can be used to calculate the truth values in a straightforward manner.

Table 2.1. Truth values in the set $T_2 = \{0, 1\}$ of negation, conjunction, disjunction, and implication.

p	q	$\overline{p}$ $1-p$	$p \wedge q$ $\min(p,q)$	$p \vee q$ $\max(p,q)$	$p \to q$ $\min(1, 1+q-p)$
1	1	0	1	1	1
1	0	0	0	1	0
0	1	1	0	1	1
0	0	1	0	0	1

Tautology

Tautology is a compound proposition form that is *true* under all possible truth values for its simple propositions.

Contradiction

Contradiction or *fallacy* is a compound proposition form that is *false* under all possible truth values for its simple propositions.

Example 2.2

The truth values for the proposition forms $p \wedge \overline{p}$ and $p \vee \overline{p}$ are presented on Table 2.2.

Table 2.2. Truth values for $p \wedge \overline{p}$ and $p \vee \overline{p}$.

p	$\overline{p}$	$p \wedge \overline{p}$	$p \vee \overline{p}$
1	0	0	1
0	1	0	1

Hence $p \wedge \overline{p}$ with truth value 0 is a *contradiction* (it is called *law of contradiction*), while $p \vee \overline{p}$ with truth value 1 is a *tautology* (it is called the *law of excluded middle: every proposition is either true or false*).

□

The branch of classical logic dealing with compound propositions is known as *propositional calculus*. Its extension is the *predicate calculus*.

Predicate

Predicate is a declarative sentence containing one or more variables or unknowns. A predicate is neither true nor false, hence it is not a proposition. Predicates are denoted by $p(x), q(x, y), \cdots$, where $x, y, \cdots$ are unknowns; they are called also *logical functions*. If in a predicate numbers are substituted for variables, the predicate becomes a proposition. For instance sentence (c) in Example 2.1 is a predicate. If x is substituted by a number, say 150, then (c) reduces to a proposition. Hence predicates are closely related to propositions; they can be considered as generalized propositions or indefinite propositions.

Correspondence between the classical logic and set theory

There is a correspondence between the logical connectives *and*, *or*, *not*, *implication* and the set operations *intersection*, *union*, *complement*, *inclusion* (subset), correspondingly, expressed in Table 2.3

It is established that this correspondence (called *isomorphism*) guarantees that every theorem or result in set theory has a counterpart in two-valued logic and vice versa. They can be obtained from one another by exchanging the corresponding symbols given in Table 2.3.

Table 2.3. Correspondence between logical connectives and set operations.

Logic	Set theory
$\vee$	$\cup$
$\wedge$	$\cap$
$^{-}$	$^{-}$
$\rightarrow$	$\subseteq$

2.2 Many-Valued Logic

Since the time when in logic the principle *every proposition is either true or false* has been declared, there have always been some doubts about it. One reason for questioning the above principle is the difficulty arising with estimating truth values of propositions expressing future events, for instance *tomorrow will rain.*[4] Future events are not yet true or false. Their truth value is unknown; it will be determined when the events happen. The classical (two-valued) logic is not sufficient to describe the truth value of these type of events. Hence it looks natural to allow a third truth value other than pure truth or falsity which leads to a three-valued logic. Depending on how the third value is defined, several three-valued logics were introduced.

Here we discuss the three-valued logic[5] proposed by Łukasiewicz (1920).

Suppose that a proposition has three truth values: *true* denoted by 1, *false* denoted by 0, and *neutral* or *indeterminate* denoted by $\frac{1}{2}$. They form the truth value set

$$T_3 = \{0, \frac{1}{2}, 1\}.$$

If p and q are propositions, the logical connectives *negation* ($^{-}$), *conjunction* ($\wedge$), *disjunction* ($\vee$), and *implication* ($\rightarrow$) are defined as in classical logic by (2.1)–(2.4) with the difference that the truth values of p and q belong to T_3.

The truth values of (2.1)–(2.4) with T_3 are given in Table 2.4.

Table 2.4. Truth values in T_3 for negation, conjunction, disjunction, implication.

p	q	$\overline{p}$	$\overline{q}$	$p \wedge q$	$p \vee q$	$p \rightarrow q$
1	1	0	0	1	1	1
1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
1	0	0	1	0	1	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	0	0	1	1
0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1
0	0	1	1	0	0	1

Example 2.3

Let us construct the truth table for the compound propositions $p \wedge \overline{p}$ and $p \vee \overline{p}$. The result is presented on Table 2.5.

Table 2.5. Truth values in T_3 for $p \wedge \overline{p}$ and $p \vee \overline{p}$.

p	$\overline{p}$	$p \wedge \overline{p}$	$p \vee \overline{p}$
1	0	0	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	0	1

Since the value $\frac{1}{2}$ appears in the third and forth columns in Table 2.5, unlike the two-valued logic (see Table 2.3), $p \wedge \overline{p}$ and $p \vee \overline{p}$, respectively, *do not satisfy the law of contradiction and the law of excluded middle.*

□

On the basis of Example 2.3 we may say that $p \wedge \overline{p}$ expresses a more general law of *quasi-contradiction*; $p \vee \overline{p}$ is a *quasi-tautology.*

The three-valued logic is a generalization of the two-valued logic. If the rows in which the truth value $\frac{1}{2}$ appears are removed from Table 2.4, then the result will be Table 2.1.

A further generalization allows a proposition to have more than three truth values. If for any given natural number $n \geq 3$, the truth values are represented by rational numbers in the interval [0, 1] that subdivide [0, 1] into equal parts, then they form the truth set T_n,

$$T_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1\}.$$

In the Łukasiewicz n-valued logic the formulas (2.1)–(2.4) for logical connectives remain valid provided that p and q are substituted by their truth values in T_n.

If the truth values are represented by all real numbers in [0, 1], i.e. the truth set is $T_\infty = [0, 1]$, the many-valued logic[6] is called *infinite-valued logic*; it is referred as the *standard Łukasiewicz logic*. There is a correspondence (isomorphism) between the fuzzy set theory and the infinite-valued logic. Complementation (1.14), intersection (1.15), and union (1.16) in fuzzy sets correspond respectively to negation (2.1), conjunction (2.2), and disjunction (2.2) in the infinite-valued logic provided that p and q are substituted by their truth values from T_∞.

2.3 What is Fuzzy Logic?

The founder of fuzzy logic is Lotfi Zadeh (1973, 1975, 1976, 1978, 1983). He made significant advancement in the establishment of fuzzy logic as a scientific discipline.

There is not a unique system of knowledge called fuzzy logic but a variety of methodologies proposing logical consideration of imperfect and vague knowledge. It is an active area of research with some topics still under discussion and debate.

We have seen that there is a correspondence (isomorphism) between classical sets and classical logic (Table 2.4).

Fuzzy sets are a generalization of classical sets and infinite-valued logic is a generalization of classical logic. There is also a correspondence (isomorphism) between these two areas (Section 2.2).

Fuzzy logic uses as a major tool—fuzzy set theory. Basic mathematical ideas for fuzzy logic evolve from the infinite-valued logic, thus there is a link between both logics. Fuzzy logic can be considered as an extension of infinite-valued logic in the sense of incorporating fuzzy sets and fuzzy relations into the system of infinite-valued logic.[7]

Fuzzy logic focuses on linguistic variables in natural language and aims to provide foundations for approximate reasoning with imprecise propositions. It reflects both the rightness and vagueness of natural language in common-sense reasoning.

The relations between classical sets, classical logic, fuzzy sets (in particular fuzzy numbers), infinite-valued logic, and fuzzy logic are schematically shown on Fig. 2.1.

Major parts of fuzzy logic deal with linguistic variables and linguistic modifiers, propositional fuzzy logic, inferential rules, and approximate reasoning.

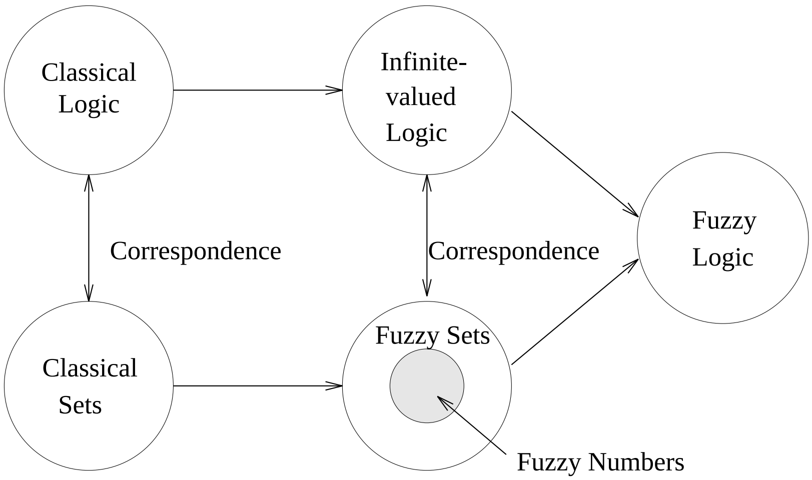

Fig. 2.1. Evolvement of Fuzzy Logic.

2.4 Linguistic Variables

Variables whose values are words or sentences in natural or artificial languages are called *linguistic variables.*

To illustrate the concept of *linguistic variable* consider the word *age* in a natural language; it is a summary of the experience of enormously large number of individuals; it cannot be characterized precisely. Employing fuzzy sets (usually fuzzy numbers), we can describe *age* approximately. *Age* is a *linguistic variable* whose values are words like *very* *young*, *young*, *middle age*, *old*, *very old*. They are called *terms* or *labels* of the linguistic variable *age* and are expressed by fuzzy sets on a universal set $U \subset R_+$ called also *operating domain* measured in years. It represents the *base variable* age. Each term is defined by an appropriate membership function. Good candidates for membership functions are triangular, trapezoidal, or bell-type shapes, without or with a flat, or parts of these (Chapter 1, Sections 1.4–1.6).

Example 2.4

Let us describe the linguistic variable *age* on the universal set $U = [0, 100]$ or operating domain of x (base variable) representing age in years (see Fig. 2.2) by triangular and part of trapezoidal numbers which specify the terms very young, young, middle age, old, and very old.

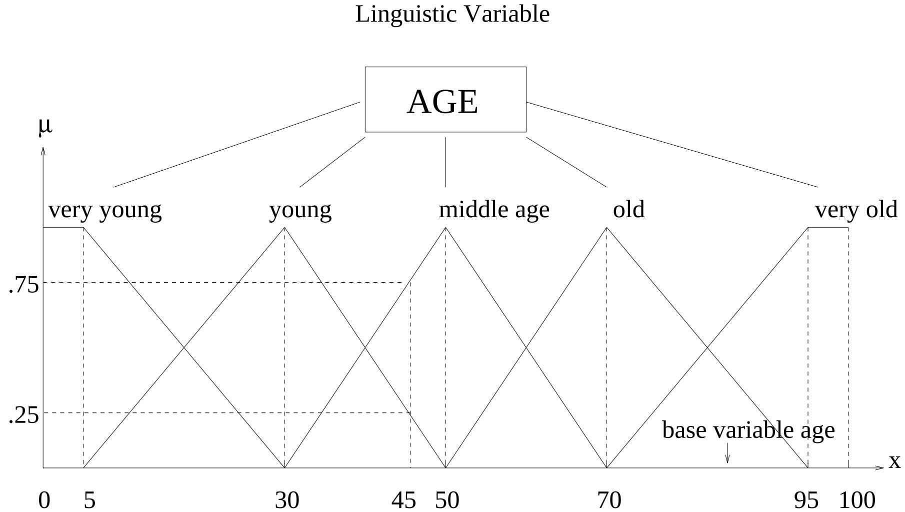

Fig. 2.2. Terms of the linguistic variable *age*.

The membership functions of the terms are:

$$\mu_{very\ young}(x) = \begin{cases} 1 & \text{for} \quad 0 \le x \le 5, \\ \frac{30-x}{25} & \text{for} \quad 5 \le x \le 30, \end{cases}$$

$$\mu_{young}(x) = \begin{cases} \frac{x-5}{25} & \text{for} \quad 5 \le x \le 30, \\ \frac{50-x}{20} & \text{for} \quad 30 \le x \le 50, \end{cases}$$

$$\mu_{middle\ age}(x) = \begin{cases} \frac{x-30}{20} & \text{for} \quad 30 \le x \le 50, \\ \frac{70-x}{20} & \text{for} \quad 50 \le x \le 70, \end{cases}$$

$$\mu_{old}(x) = \begin{cases} \frac{x-50}{20} & \text{for} \quad 50 \le x \le 70, \\ \frac{95-x}{25} & \text{for} \quad 70 \le x \le 95, \end{cases}$$

$$\mu_{very\ old}(x) = \begin{cases} \frac{x-70}{25} & \text{for} \quad 70 \le x \le 95, \\ 1 & \text{for} \quad 95 \le x \le 100. \end{cases}$$

For instance, a person whose age is 45 is young to degree 0.25 and middle age to degree 0.75. The degrees are found by substituting 45 for x into the second equation of the term $\mu_{young}(x)$ and first equation of the term $\mu_{middleage}(x)$, correspondingly. Hence a person whose age is 45 is less *young* (degree 0.25) and more *middle age* (degree 0.75).

□

Linguistic variables play an important role in applications and in particular in financial and management systems. For example, *truth*,[8] *confidence*, *stress*, *income*, *profit*, *inflation*, *risk*, *investment*, etc. can be understood to be linguistic variables.

2.5 Linguistic Modifiers

Let $x \in U$ and $\mathcal{A}$ is a fuzzy set with membership function $\mu_{\mathcal{A}}(x)$. We denote by m a *linguistic modifier*, for instance *very*, *not*, *fairly* (more or less), etc. Then by $m\mathcal{A}$ we mean a modified fuzzy set by m with membership function $\mu_{m\mathcal{A}}(x)$.

The following selections for $\mu_{m\mathcal{A}}(x)$ are often used to describe the modifiers *not, very,* and *fairly*:

$$\text{not,} \quad \mu_{not\mathcal{A}}(x) = 1 - \mu_{\mathcal{A}}(x), \tag{2.5}$$

$$\text{very,} \quad \mu_{very\mathcal{A}}(x) = [\mu_{\mathcal{A}}(x)]^2, \tag{2.6}$$

$$\text{fairly,} \quad \mu_{fairly\mathcal{A}}(x) = [\mu_{\mathcal{A}}(x)]^{\frac{1}{2}}. \tag{2.7}$$

Example 2.5

Consider the fuzzy set $\mathcal{A}$ describing the linguistic value *high score* (*high*) related to a loan scoring model defined as

x	0	20	40	60	80	100
$\mu_{high}(x)$	0	0.2	0.5	0.8	0.9	1

where x is a base variable over $U_1 = \{0, 20, 40, 60, 80, 100\}$, the universal set; it is numerical in nature and represents a discrete scale of the scores used in the model.

The graph of $\mu_{high}(x)$ is shown in Fig. 2.3. by dots.

The linguistic value *high score* can be modified to become *not high score, very high score,* and *fairly high score* by using (2.5)–(2.7). First let us find *not high score*:

$$\mu_{not\ high}(x) = 1 - \mu_{high}(x).$$

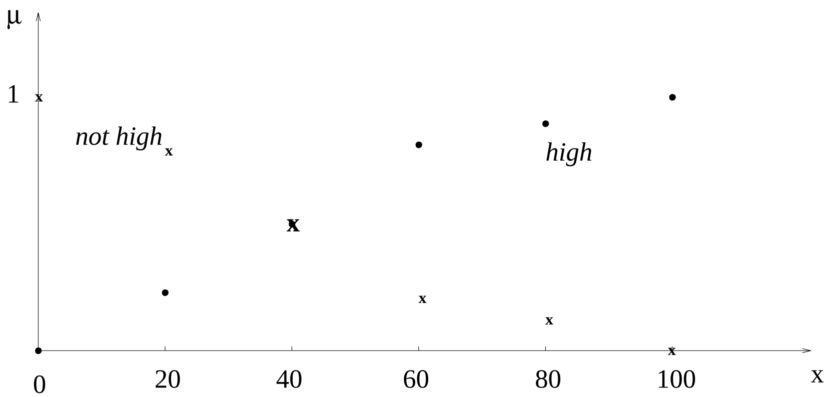

Fig. 2.3. Fuzzy sets *high score* (dots) and *not high score* (crosses).

Using the table for $\mu_{high}(x)$ we calculate

$$\begin{aligned}
\mu_{not\ high}(0) &= 1 - \mu_{high}(0) = 1 - 0 = 1,\\
\mu_{not\ high}(20) &= 1 - \mu_{high}(20) = 1 - 0.2 = 0.8,\\
\mu_{not\ high}(40) &= 1 - \mu_{high}(40) = 1 - 0.5 = 0.5,\\
\mu_{not\ high}(60) &= 1 - \mu_{high}(60) = 1 - 0.8 = 0.2,\\
\mu_{not\ high}(80) &= 1 - \mu_{high}(80) = 1 - 0.9 = 0.1,\\
\mu_{not\ high}(100) &= 1 - \mu_{high}(100) = 1 - 1 = 0.
\end{aligned}$$

Hence for the fuzzy set *not high score* we obtain the table (see Fig. 2.3)

x	0	20	40	60	80	100
$\mu_{not\ high}(x)$	1	0.8	0.5	0.2	0.1	0

Similarly we construct the tables for the fuzzy sets *very high score* and *fairly high score*. The results are presented in Fig. 2.4.

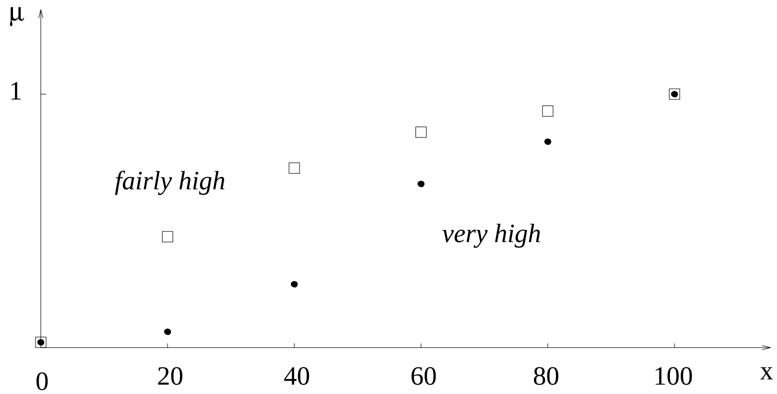

Fig. 2.4. Fuzzy sets *very high score* (dots) and *fairly high score* (squares).

$$\mu_{very\ high}(x) = [\mu_{high}(x)]^2.$$

x	0	20	40	60	80	100
$\mu_{very\ high}(x)$	0	0.04	0.25	0.64	0.81	1

$$\mu_{fairly\ high}(x) = [\mu_{fast}(x)]^{\frac{1}{2}}.$$

x	0	20	40	60	80	100
$\mu_{fairly\ high}(x)$	0	0.447	0.707	0.894	0.949	1

□

Example 2.6

The fuzzy set $\mathcal{B}$ describes the linguistic value *good credit* (*good*). The membership function of $\mathcal{B}$ is (see Fig. 2.5)

y	0	20	40	60	80	100
$\mu_{good}(y)$	0	0.2	0.4	0.7	1	1

where y is a base variable over $U_2 = \{0, 20, 40, 60, 80, 100\}$, the universal set; it is a discrete scale for credit rating similar to that in Example 2.5 concerning *high score*.

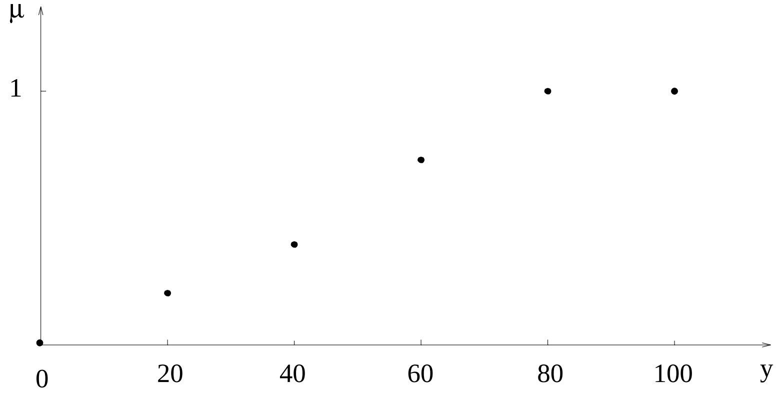

Fig. 2.5. Fuzzy set *good credit*.

Following Example 2.5 we modify *good credit* using (2.5)–(2.7). The results are given below.

y	0	20	40	60	80	100
$\mu_{not\ good}(y)$	1	0.8	0.6	0.3	0	0
$\mu_{very\ good}(y)$	0	0.04	0.16	0.49	1	1
$\mu_{fairly\ good}(y)$	0	0.45	0.63	0.84	1	1

□

The representation of $m\mathcal{A}$ should express the meaning of the linguistic modifier adequately. However there is no unique way to do this.

For instance the modifier *very* described by (2.6) can be expressed differently by a shift of the membership function $\mu_{\mathcal{A}}(x)$ to the right,

$$\mu_{very\mathcal{A}}(x) = \mu_{\mathcal{A}}(x - c), \quad a + c \leq x \leq b + c,$$

where $c > 0$ is a suitable constant (Fig. 2.6). Similarly *fairly* can be described by a shift of $\mu_{\mathcal{A}}(x)$ to the left.

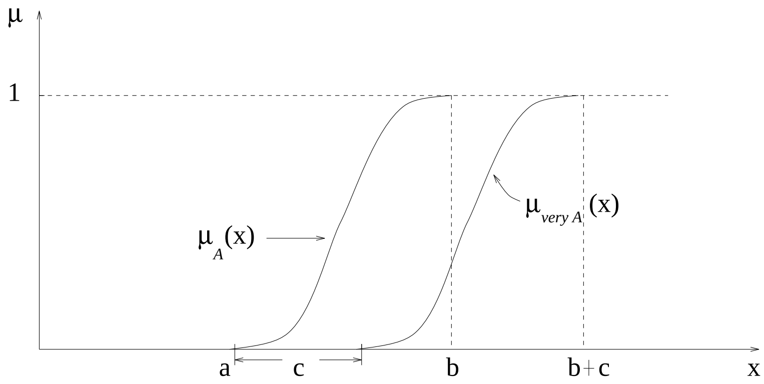

Fig. 2.6. Modifier *very* expressed by a shift.

Also $\mu_{\mathcal{A}}(x)$ and $\mu_{very\ \mathcal{A}}(x)$ can be defined as terms of a linguistic variable; this was already demonstrated in Example 2.1, Fig. 2.2 (*old* and *very old, young* and *very young*).

2.6 Composition Rules for Fuzzy Propositions

In two-valued logic a proposition p is true or false (Section 2.1). In many-valued logic and fuzzy logic the concept of proposition is considered in a broader context, i.e. a proposition is *true to a degree* in the interval [0, 1]. The truth of a proposition p in fuzzy logic is expressed by a fuzzy set, hence by its membership function.

Below are listed some important propositions involving the fuzzy sets $\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x))\}$ and $\mathcal{B} = \{(y, \mu_{\mathcal{B}}(y))\}$.

(i) x is $\mathcal{A}$, proposition in *canonical form*;

(ii) x is $m\mathcal{A}$, *modified* proposition;

(iii) If x is $\mathcal{A}$ then y is $\mathcal{B}$, *conditional* proposition.

The propositions (i)–(iii) are illustrated in the following example.

Example 2.7

Let *high score* and *good credit* be described by the fuzzy sets defined in Examples 2.5 and 2.6.

(i) Client loan score is *high score* (canonical form).

(ii) Client loan score is a *very high score* (modified proposition).

(iii) If client loan score is *high score* then client loan credit is *good credit* (conditional proposition).

□

Operation *composition* consists of two propositions p and q joined by logical connectives.

The propositions are defined by

$$p \stackrel{\triangle}{=} x \text{ is } \mathcal{A}, \qquad q \stackrel{\triangle}{=} y \text{ is } \mathcal{B}, \tag{2.8}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the fuzzy sets (see Fig. 2.7)

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x))|x \in A \subset U_1\}, \quad \mathcal{B} = \{(y, \mu_{\mathcal{B}}(y))|y \in B \subset U_2\}. \tag{2.9}$$

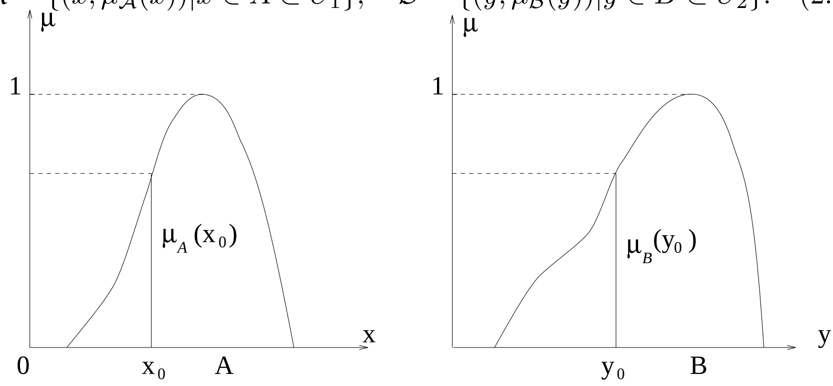

Fig. 2.7. Truth values $\mu_{\mathcal{A}}(x_0), \mu_{\mathcal{B}}(y_0)$.

We can give here the following interpretation. The membership grades $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(y)$ represent the truth values of the propositions (2.8), correspondingly. Conversely, the truth values of (2.8) are expressed by the membership functions $\mu_{\mathcal{A}}(x)$ and $\mu_{\mathcal{B}}(y)$. If x_0 and y_0 are specified values on the universes U_1 and U_2, respectively, then the truth values $\mu_{\mathcal{A}}(x_0), \mu_{\mathcal{B}}(y_0)$ of propositions x_0 is $\mathcal{A}$, y_0 is $\mathcal{B}$ are shown in Fig. 2.7 where the membership functions are assumed continuous.

Composition conjunction $p \wedge q$

The truth value (tr) of $p \wedge q$ (p *and* q) is defined by

$$\mathrm{tr}(p \wedge q) = \mu_{\mathcal{A} \dot{\times} \mathcal{B}}(x, y) = \min(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(y)), (x, y) \in A \times B, \quad (2.10)$$

where $\mu_{\mathcal{A} \dot{\times} \mathcal{B}}(x, y)$ is the membership function of the direct min product (Section 1.8 (1.21)).

Composition disjunction $p \vee q$

The truth value of $p \vee q$ (p *or* q) is defined by

$$\mathrm{tr}(p \vee q) = \mu_{\mathcal{A} \dot{\times} \mathcal{B}}(x, y) = \max(\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(y)), (x, y) \in A \times B, \quad (2.11)$$

where $\mu_{\mathcal{A} \dot{\times} \mathcal{B}}(x, y)$ is the membership function of the direct max product (Section 1.8 (1.22)).

Composition implication $p \to q$

The truth value of $p \to q$ (*if* p ... *then* q) is defined by

$$\mathrm{tr}(p \to q) = \min(1, 1 - \mu_{\mathcal{A}}(x) + \mu_{\mathcal{B}}(y)), (x, y) \in A \times B, \quad (2.12)$$

meaning that to each pair (x, y) in the Cartesian product $A \times B$ we have to attach as a membership value the smaller between 1 and $1 - \mu_{\mathcal{A}}(x) + \mu_{\mathcal{B}}(y)$.

There are also several other definitions for composition implication (see for instance Mizumoto (1985)).

The rules (2.10)–(2.12) originate from the classical logic and many-valued logics of Łukasiewicz (see (2.2)–(2.4)).

The right hand sides of (2.10)–(2.12) are membership functions of *fuzzy relations* since (x, y) belongs to the Cartesian product $A \times B \subset U_1 \times U_2$. Hence the truth values of composition rules are presented by *fuzzy relations.*

In formulas (2.10)–(2.12) the notation tr which stands for truth could be omitted similarly to Chapter 1, Section 2.1.

It should be stressed that the membership functions of $\mathcal{A}$ and $\mathcal{B}$ (see 2.9) have different arguments, x and y, correspondingly. From this point of view the operations min (2.10) and max (2.11) expressing the logical connectives *and* and *or* differ from the operations min (1.9) and max (1.10) in Section 1.3.

Example 2.8

Consider two propositions p and q of the type (2.8) in canonical form defined by

$$p \triangleq x \text{ is } \textit{high score}, \qquad q \triangleq y \text{ is } \textit{good credit},$$

related to a loan scoring model where *high score* is the fuzzy set $\mathcal{A}$ in Example 2.5 defined on the universe U_1 (operating domain of x representing client loan score) and *good credit* is the fuzzy set $\mathcal{B}$ in Example 2.6, defined on the universe U_2 (operating domain of y representing client credit rating).

(i) The truth value of composition conjunction (2.10) is the membership function $\mu_{\mathcal{A}\underset{\cdot}{\times}\mathcal{B}}(x, y)$ of the relation $\mathcal{R}$ presented on Table 2.6.

Table 2.6. Truth value of x *is high score and* y *is good credit.*

		B					
	x \ y	0	20	40	60	80	100
	0	0	0	0	0	0	0
	20	0	0.2	0.2	0.2	0.2	0.2
A	40	0	0.2	0.4	0.5	0.5	0.5
	60	0	0.2	0.4	0.7	0.8	0.8
	80	0	0.2	0.4	0.7	0.9	0.9
	100	0	0.2	0.4	0.7	1	1

To construct the table we use the direct min product (2.10), i.e. consider all ordered pairs $(x_i, y_j), x_i \in A, y_j \in B$ in the Cartesian product $A \times B$ and in the cell (x_i, y_j), located at the intersection of row x_i and column y_j, write the smaller value of $\mu_{\mathcal{A}}(x_i)$ and $\mu_{\mathcal{B}}(y_j)$. For instance let us calculate the truth values in the third row in Table 2.6 when $x = 40$ and y takes the values in B:

$$\mu_{high}(40) = 0.5 > \mu_{good}(0) = 0, \quad \mu_{\mathcal{A}\underset{\cdot}{\times}\mathcal{B}}(40, 0) = 0$$
$$\mu_{high}(40) = 0.5 > \mu_{good}(20) = 0.2, \quad \mu_{\mathcal{A}\underset{\cdot}{\times}\mathcal{B}}(40, 40) = 0.2$$
$$\mu_{high}(40) = 0.5 > \mu_{good}(40) = 0.4, \quad \mu_{\mathcal{A}\underset{\cdot}{\times}\mathcal{B}}(40, 40) = 0.4$$
$$\mu_{high}(40) = 0.5 < \mu_{good}(60) = 0.7, \quad \mu_{\mathcal{A}\underset{\cdot}{\times}\mathcal{B}}(40, 60) = 0.5$$

$$\mu_{high}(40) = 0.5 < \mu_{good}(80) = 1, \quad \mu_{\mathcal{A} \dot{\times} \mathcal{B}}(40, 80) = 0.5$$
$$\mu_{high}(40) = 0.5 < \mu_{good}(100) = 1, \quad \mu_{\mathcal{A} \dot{\times} \mathcal{B}}(40, 100) = 0.5\,.$$

(ii) To find the truth value of composition disjunction (2.11) we use the direct max product and proceed like in case (i) with the only difference that in the cell (x_i, y_i) we write the larger value of $\mu_{\mathcal{A}}(x_i)$ and $\mu_{\mathcal{B}}(y_i)$.

(iii) To find the truth value of composition implication (2.12) for each pair $(x_i, y_j) \in A \times B$ we calculate $1 - \mu_{\mathcal{A}}(x_i) + \mu_{\mathcal{B}}(y_j)$ and then take this value if it is smaller than 1; otherwise we take 1.

□

2.7 Semantic Entailment

Semantic entailment concerns *inclusion* of fuzzy sets taking part in propositions. Consider the propositions

$$p \triangleq x \text{ is } \mathcal{A}, \qquad q \triangleq x \text{ is } \mathcal{B},$$

both defined on the same universe U. We say that proposition p *semantically entails* proposition q (or q is *semantically entailed* by p), denoted by

$$p \to q \tag{2.13}$$

if and only if

$$\mu_A(x) \le \mu_B(x), \quad x \in U. \tag{2.14}$$

The meaning of (2.13), based upon the concept of subset (2.14) introduced in Section 1.3, is that p brings as an inevitable consequence q in the sense that q is less specific than p.

Example 2.9

The proposition

$$p \triangleq \textit{Client loan score is a very high score}$$

semantically entails the proposition

$$q \triangleq \textit{Client loan score is a high score}$$

no matter how the linguistic variable *high score* is defined. Hence from the proposition *Client loan score is a very high score* we may infer that *Client loan score is a high score.* We say that the *semantic entailment is strong.*

To be more specific assume that *high* and *very high* are defined as they appear in Examples 2.5 (see Figs. 2.3 and 2.4). Clearly (2.14) is satisfied since

$$\mu_{very\ high}(x) \leq \mu_{high}(x).$$

□

Example 2.10

The proposition

$$p \triangleq \textit{Client loan score is not a high score}$$

may or may not *semantically entail* the proposition

$$q \triangleq \textit{Client loan score is a low score}$$

depending on how the fuzzy sets *high* and *low* are defined. In this case we say the *semantic entailment is not strong.*

Let us assume that *not high* is defined as in Example 2.5 (Fig. 2.3) and *low* is defined below (the universe U is the same) in two slightly different ways

x	0	20	40	60	80	100
$\mu_{low}^{(1)}(x)$	1	0.85	0.6	0.3	0.2	0.1
$\mu_{low}^{(2)}(x)$	1	0.7	0.4	0.2	0.15	0.1

Clearly (see Fig. 2.8)

$$\mu_{not\ high}(x) \leq \mu_{low}^{(1)}(x), \qquad \mu_{not\ high}(x) \approx \mu_{low}^{(2)}(x),$$

hence the semantic entailment is not strong; if *low* is defined by $\mu_{low}^{(1)}(x)$, (2.14) is satisfied; if *low* is defined by $\mu_{low}^{(2)}(x)$, (2.14) is not satisfied.

From the proposition *Client loan score is not a high score* we may or may not infer that *Client loan score is a low score.*

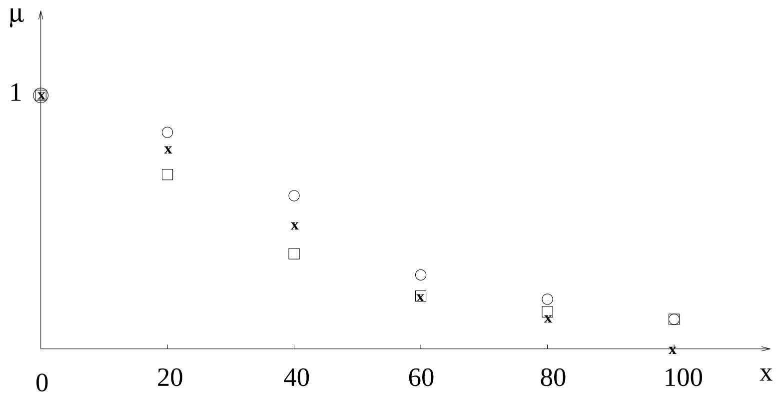

Fig. 2.8. Fuzzy sets *not high* (crosses), *low* (1) (circles), *low* (2) (squares). □

Semantic entailment plays an important role in fuzzy logic as a main *rule of inference* known as *entailment principle* in the sense that the validity of proposition q is inferred from the validity of proposition p (see (2.13)) if and only if (2.14) holds.

The entailment principle can be generalized for more that two proposition. For instance, if $p \triangleq x$ is $\mathcal{A}$, $q \triangleq x$ is $\mathcal{B}$, $r \triangleq x$ is $\mathcal{C}$, and $\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x), \mu_{\mathcal{C}}(x)$ are the corresponding membership functions, we have

$$p \to q \to r$$

if and only if

$$\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x) \leq \mu_{\mathcal{C}}(x).$$

2.8 Notes

1. Classical (two-valued) logic has its roots in the work of George Boole (1815–1864) after whom *Boolean algebra*, a branch of classical logic, is named.

   The modern two-valued logic started with the book *Begriffsschrift* (1879) by Gottlob Frege (1848–1925), for whom the meaning of logic is based on the rules for manipulating symbols and the propositional connectives *not*, *or*, *and*, *if ... then*.

Charles Peirce (1839–1914) who made important contributions to the two-valued logic in his study *On the Algebra of Logic* (1880) may be considered as one of the pioneers of many-valued logic. He wrote: "Vagueness is no more to be done away with in the world of logic than friction in mechanics."

Further advancement in two-valued logic and its use to formalize mathematics was made by Bertrand Russell (logician and philosopher) and Alfread Whitehead (mathematician and philosopher) in their fundamental work *Principia Mathematica* which appeared in three volumes between 1910–1913.

2. In order to be more precise while denoting propositions and their truth values in this Chapter we may use $tr\ p$ to express the truth value of p. Then for instance formula (2.2) will take the form

$$tr(p \wedge q) = \min(tr\ p,\ tr\ q),$$

where $tr\ p$ and $tr\ q$ belong to the set $\{0, 1\}$.

3. The truth tables were introduced by the philosopher Ludwig Wittgenstein (1889–1951) in *Tractatus Logico-Philosophicus* (1922). He made significant contributions to the philosophy of mathematics.

4. The origins of many-valued logics can be traced back to ancient Greek philosophy. Aristotle (384–322 B.C.) himself, the father of logic, made remarks about the problematic truth values of propositions expressing future events. In *Metaphysics* he wrote "*The more and less are still present in the nature of things.*"

5. The three-valued logic was established independently by J. Łukasiewicz (1920) and E. Post (1921). They also introduced many-valued logics.

6. The many-valued logic is a generalization, not a rejection, of the classical two-valued logic. The many-valued logic only dismantles the philosophical illusions about the absoluteness of classical logic and proposes a more general approach towards solving logical problems.

7. A part of fuzzy logic is *possibility theory* introduced by Zadeh (1978). The basic concept of possibility theory is that of *possibility distribution*. The membership function $\mu_{\mathcal{A}}(x)$ of a fuzzy set $\mathcal{A}$ can be considered as a constraint or restriction on the values (grades, degrees of membership) that can be assigned to $x \in U$. In other words, the degree of membership $\mu \in [0,1]$ is interpreted as a possibility level $\pi \in [0,1]$. The fuzzy set $\mathcal{A}$ is interpreted as a possibility distribution $\Pi(x)$; to the membership function $\mu_{\mathcal{A}}(x)$ corresponds the function $\pi(x)$ describing the possibility distribution $\Pi(x)$; $\pi(x) \in [0,1]$; actually $\pi(x) = \mu_{\mathcal{A}}(x)$.

8. Perhaps the most important linguistic variable is *truth*. It is described by a fuzzy set with membership function $\mu_{true}(x), \mu \in [0,1]$ (we are using *true* instead of *truth*). *False* is interpreted as *not true*.

   Truth and its terms have been defined differently in fuzzy logic. We consider first the simplest definition introduced by Baldwin (1979)

   $$true \triangleq \{(x, \mu_{true}(x)) \mid x \in [0,1], \mu_{true}(x) = x, \mu \in [0,1]\}.$$

   The modifiers (2.5)–(2.7) applied to $\mu_{true}(x) = x$ give that

   $$\mu_{not\ true}(x) = \mu_{false}(x) = 1 - x,$$
   $$\mu_{very\ true}(x) = [\mu_{true}(x)]^2 = x^2,$$
   $$\mu_{fairly\ true}(x) = [\mu_{true}(x)]^{\frac{1}{2}} = x^{\frac{1}{2}}.$$

   Similarly one can define

   $$\mu_{very\ false}(x) = (1-x)^2, \quad \mu_{fairly\ false}(x) = (1-x)^{\frac{1}{2}}.$$

   The extreme case $x = 1$ in $\mu_{true}(x) = x$ gives the singleton $\mu_{absolute\ true}(1) = 1$; then it follows that $\mu_{absolute\ false}(0) = 1$.

   The linguistic variables *truth* and *false* are shown in Fig. 2.9. On the same figure are shown also their modifications and the modified modifications:

   $$\mu_{very\ very\ true}(x) = [\mu_{very\ true}(x)]^2 = x^4,$$
   $$\mu_{very\ very\ false}(x) = [\mu_{very\ false}(x)]^2 = (1-x)^4.$$

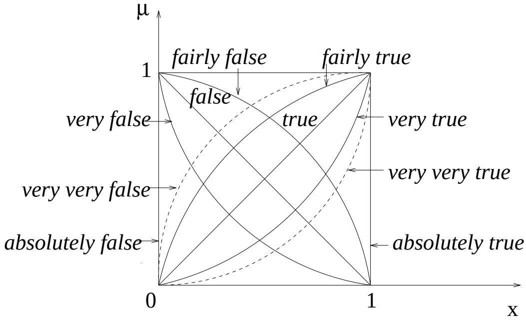

Fig. 2.9. Linguistic variable *truth* and various modifications.

Zadeh (1975) defined *truth* by the membership function (Fig. 2.10)

$$\mu_{true}(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le a, \\ 2(\frac{x-a}{1-a})^2 & \text{for} \quad a \le x \le \frac{a+1}{2}, \\ 1 - (\frac{x-1}{1-a})^2 & \text{for} \quad \frac{a+1}{2} \le x \le 1. \end{cases}$$

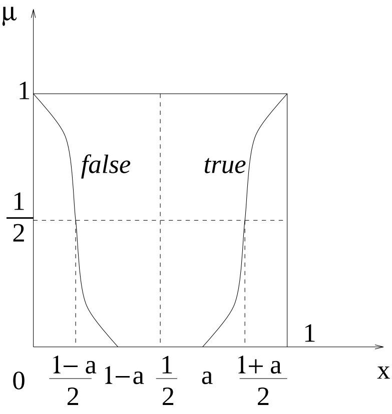

Fig. 2.10. Linguistic variable *truth* (Zadeh).

Here $1 + \frac{a}{2}$ is the crossover point. The parameter $a \in [0, 1]$ indicates the subjective selection of the minimum value of a in such a way that for $x > a$ the degree of truth is positive, i.e. $\mu_{true}(a) > 0$. The membership function of *false* is defined by $\mu_{false}(x) = \mu_{true}(1-x)$. The terms $\mu_{very\ true}(x)$ and $\mu_{fairly\ true}(x)$ can be calculated from (2.6) and (2.7).